Math 432: Set Theory and Topology HOMEWORK 2 Due date: Feb 2 (Thu)

Exercises from Kaplansky's book.

Sec 1.5: 2, 4

Sec 1.3: 4, 7, 10

Other (mandatory) **exercises.**

1. Determine which of the following E are equivalence relations and, for those which are, describe all equivalence classes. Prove your answers.

(a) On $\mathbb{R}$: $xEy :\Leftrightarrow |x-y| < 1$.

(b) On $\mathbb{R}$: $xEy :\Leftrightarrow$ there is no integer in the interval $\big(\min(x,y), \max(x,y)\big]$.

(c) On $\mathbb{R}$: $xEy :\Leftrightarrow$ either $x = y$ or one of x, y is positive.

(d) On $\mathbb{R}$: $xEy :\Leftrightarrow$ either $x = y$ or both of x, y are positive.

(e) On $\mathbb{R}^2$: $(x_0, x_1)E(y_0, y_1) :\Leftrightarrow x_0 = y_0$.

(f) On $\mathscr{P}(X)$, for some set X: $A\,E\,B :\Leftrightarrow$ there is a bijection $A \to B$.

2. Prove that every finite partially ordered set has a maximal and a minimal elements.

HINT: Arguing by contradiction, build an infinite subset.

3. Let $(X, \leq)$ be a partially ordered set in which **every subset** has a top (= maximum) and a bottom (= minimum) elements.

(a) Prove that $\leq$ is a total order.

(b) Let x_0 be the bottom element of X, let x_1 be the bottom element of $X - \{x_0\}$, let x_2 be the bottom element of $X - \{x_0, x_1\}$, ... let x_{n+1} be the bottom element of $X - \{x_0, x_1, \dots, x_n\}$, and so on ... Prove that this process must terminate at some finite stage n.

(c) Conclude the following theorem.

Theorem. *If in a partially ordered set every subset has a top and a bottom elements, then X is finite.*

4. Let $(X, \leq)$ be a partially ordered set in which the largest length of a subchain[1] is n. Let $\mathcal{C}$ be the collection of all subchains of X that have n elements (the chains in $\mathcal{C}$ may overlap and this is ok); in particular, each $C \in \mathcal{C}$ is a chain, whence has a top element, denoted by $\max C$.

(a) Prove that the set $M := \{\max C : C \in \mathcal{C}\}$ is an antichain.

(b) Prove that the largest length of a subchain of $X \setminus M$ is $n - 1$.

(c) Using the previous two parts and induction, prove the following theorem.

Theorem. *If, in a partially ordered set $(X, \leq)$, the largest length of a subchain is n, then X is a union of n antichains and n is the least integer with this property.*

[1] In a partially ordered set $(X, \leq)$, *subchain* is just a subset $A \subseteq X$ such that $(A, \leq)$ is a chain.